So, $v \in L(S)$.

Conversely, Let $v \in L(S)$. Then $\exists$ $v_1, v_2, \ldots, v_m$ in S and scalars $k_1, k_2, \ldots k_m$ such that

$$v = \sum_{i=1}^{m} k_i v_i$$

Hence $$\sum_{i=1}^{m} k_i v_i + (-1)v = \theta \quad \text{——} \quad (1)$$

Since, $v \neq v_i$ then $\{v_1, v_2, \ldots, v_n, v\}$ is linearly dependent. So, $S \cup \{v\}$ is linearly dependent as it is a superset of $\{v_1, v_2, \ldots, v_4, v\}$.

2.3 Dimension of a subspace

Our next result relates the dimension of a subspace to the dimension of the vector space that contains it.

Theorem 2.3.1 Let W be a subspace of a finite dimensional vector space V. Then W is finite dimensional and $\dim W \leq \dim V$. Moreover if $\dim W = \dim V$, Then $V = W$

Proof: Let $\dim V = n$. If $W = \{\theta\}$, then W is finite dimensional and $\dim W = 0 \leq n$. Otherwise, W contains a non-zero vector α_1; so $\{\alpha_1\}$ is a linearly independent set. Continue choosing vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$ in W such that $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is linearly independent. Since no linearly independent subset of V can contain more than n vectors, this process must stop at a stage where $k \leq n$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is linearly independent but adjoining any other vector from W

produces a linearly dependent set. Theorem 2.2.8 implies that $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ generates W and hence it is a basis of W. So, $\dim W = k \leq n$.

If $\dim W = n$, then a basis for W is a linearly independent ~~set~~ subset of V containing n vectors. So, Theorem 2.2.6 implies that this basis for W is also a basis for V. So, $W = V$

Theorem 2.3.2 Let V be a vector space over a field F and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a subset of V. Then S is a basis of V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of S, that is, v can be expressed in the form

$$v = \sum_{i=1}^{n} a_i \alpha_i, \quad \text{for unique } a_1, a_2, \ldots, a_n \in F$$

Proof: As S is a basis, $L(S) = V$. ~~So $v \in V$~~ ~~a~~ So, for $v \in V$, v is a linear combination of vectors of S. Suppose that $v = \sum a_i \alpha_i = \sum b_i \alpha_i$, $a_1, \ldots, a_n \in F$, $b_1, \ldots, b_n \in F$ are two representations of v. So, $\sum (a_i - b_i)\alpha_i = \theta$. As S is linearly independent $a_i - b_i = 0$ for $i = 1, 2, \ldots, n$

So, $a_i = b_i$, for $i = 1, \ldots, n$. So, v is uniquely expressed as a linear combination of the vector of S. Conversely, let every vector of V is uniquely expressed as linear combination of vectors of S.

Let $\sum_{i=1}^{n} c_i \alpha_i = \theta$. So, $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \theta$

So, $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0.\alpha_1 + 0.\alpha_2 + \cdots + 0.\alpha_n$

So, As the representation of a vector in V is unique

$c_1 = 0, c_2 = 0, \ldots, c_n = 0$. So, S is linearly independent

As every $v \in V$, can be expressed uniquely as a linear combination of vectors of S, so $L(S) = V$

Hence S is a basis of V.

Some examples of subspaces whose dimensions is determined here.

Example 1 Let $W = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$. We can easily check that W is a subspace of $\mathbb{R}^3$.

Let $(a,b,c) \in W \Rightarrow a+b+c = 0$ and $a,b,c \in \mathbb{R}$

So, $(a,b,c) = (a,b,-a-b) = a(1,0,-1) + b(0,1,-1)$

Let $S = \{(1,0,-1),(0,1,-1)\}$. Then S is a basis of W. So, $\dim(W) = 2$

Example 2 The set of diagonal $n\times n$ matrices W is a subspace of $M_{n\times n}(F)$. A basis for W is

$\{E_{11}, E_{22}, \ldots, E_{nn}\}$ where E_{ij} is the matrix whose only non-zero element is the ij th element. So, $\dim(W) = n$

Example 3 We can check that (check it) that the set of symmetric $n\times n$ matrices is a subspace W of $M_{n\times n}(F)$. A basis for W is

the set $\{A_{ij} : i = 1,2,\ldots,n,\ j = 1,2,\ldots,n\}$ and $i \le j\}$

where A_{ij} is the $n\times n$ matrix having

ijth element and jith element as 1 and 0 elsewhere.
So, it follows that

$$\dim(W) = n + (n-1) + \cdots 1 = \frac{n(n+1)}{2}$$

Corollary of Theorem 2.3.1 If W is a subspace of a finite dimensional vector space V, then any basis of W can be extended to a basis of V.

Proof: Exercise.

Example: The set of all polynomials of the form

$$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_0 \quad \text{where}$$

$a_5, a_4, a_3, a_2, a_0 \in F$, is a subspace W of $P_5(\mathbb{R})$. A basis for W is $\{1, x^2, x^3, x^4, x^5\}$ which is a subset of the standard basis of $P_5(\mathbb{R})$

2.4 Subspaces of $\mathbb{R}^n$: From Theorem 2.3.1, we can determine the subspaces of $\mathbb{R}^n$. They are of dimensions $0, 1, 2, \ldots, n$. $\{\theta\}$ is of dimension 0 and $\mathbb{R}^n$ is of dimension n. We ~~can~~ consider the cases of $\mathbb{R}^3$ and $\mathbb{R}^2$ in particular. The subspaces of $\mathbb{R}^2$ are of dimension 0, 1 and 2. $\{(0,0)\}$ is of dimension zero and $\mathbb{R}^2$ is of dimension. Any subspace of $\mathbb{R}^2$ having dimension 1 consists of all scalar ~~mult~~ multiples of some non-zero vector (a, b). If $b=0, a\neq 0$ then the subspace is geometrically the x-axis,