**Corollary:** If $f:[a,b]\to\mathbb{R}$ be piecewise continuous on $[a,b]$ then $f$ is integrable on $[a,b]$.

**Theorem 1.6.4** Let $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$ and let $f$ be continuous on $[a,b]$ except on a infinite subset $S\subset[a,b]$ such that the number of limit points of $S$ is finite. Then $f$ is integrable on $[a,b]$.

**Proof:** Since $f$ is bounded on $[a,b]$, $\exists$ a positive number $k$ such that $|f(x)|<k$ for all $x\in[a,b]$.

Let $S'$ (the derived set of $S$) $=\{x_1,x_2,\dots,x_m\}$ such that $x_1<x_2<\dots<x_m$

**Case 1** Let $a<x_1<x_2<\dots<x_m<b$

Let us choose $\varepsilon>0$. Let the points $x_1,x_2,\dots,x_m$ be enclosed by $m$ non-overlapping subintervals $\left[x_1-\frac{\delta_1}{2},x_1+\frac{\delta_1}{2}\right],\left[x_2-\frac{\delta_2}{2},x_2+\frac{\delta_2}{2}\right],\dots,\left[x_m-\frac{\delta_m}{2},x_m+\frac{\delta_m}{2}\right]$ of $[a,b]$ such that $a<x_1-\frac{\delta_1}{2}$, $b>x_m+\frac{\delta_m}{2}$ and

$\delta_1+\delta_2+\dots+\delta_m<\frac{\varepsilon}{4K}$

Let $M^{(r)}=\sup\limits_{x\in\left[x_r-\frac{\delta_r}{2},\,x_r+\frac{\delta_r}{2}\right]} f(x)$, $m^{(r)}=\inf\limits_{x\in\left[x_r-\frac{\delta_r}{2},\,x_r+\frac{\delta_r}{2}\right]} f(x)$,

for $r=1,2,\dots,m$

Then $M^{(r)}-m^{(r)}<2K$, for $r=1,2,\dots,m$

On each of the remaining $m+1$ subintervals $\left[a,x_1-\frac{\delta_1}{2}\right]$,

$\left[x_1+\frac{\delta_1}{2}, x_2-\frac{\delta_2}{2}\right], \cdots \left[x_m+\frac{\delta_m}{2}, b\right]$, f is continuous except for a finite number of points. So, f is integrable on each of these intervals, by Theorem 1.6.4.

Therefore $\exists$ partitions $P_1$ of $\left[a, x_1-\frac{\delta_1}{2}\right]$, $P_2$ of $\left[x_1+\frac{\delta_1}{2}, x_2-\frac{\delta_2}{2}\right]$, $\cdots, P_{m+1}$ of $\left[x_m+\frac{\delta_m}{2}, b\right]$ such that

$$U(P_k, f) - L(P_k, f) < \frac{\varepsilon}{2(m+1)}, \text{ for } k=1,2,\cdots m+1$$

The partitions $P_1, P_2, \cdots, P_{m+1}$ are disjoint

Let $P = P_1 \cup P_2 \cdots \cup P_{m+1}$. Then P is a partition of $[a, b]$.

$$U(P,f)-L(P,f) = \sum_{k=1}^{m+1}\left(U(P_k,f)-L(P_k,f)\right) + \sum_{r=1}^{m}\left(M^{(r)}-m^{(r)}\right)\delta_r$$

$$< \frac{\varepsilon}{2(m+1)}\cdot(m+1) + 2k(\delta_1+\delta_2+\cdots+\delta_m) < \varepsilon$$

Thus for a chosen $\varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

This being a sufficient condition for integrability, f is integrable on $[a, b]$.

<u>Case 2</u> Either $a = x_1$ or $x_m = b$ or both

If $a = x_1$, the subinterval enclosing the point $x_1$ can be chosen as $[a, a+\delta_1]$. If $x_m = b$, the last subintervals can be chosen as $[b-\delta_m, b]$.

In any case, proceeding with similar arguments it can be proved that $f$ is integrable on $[a,b]$

This completes the proof.

Note. Since the set $S$ is bounded and infinite, the derived set can not be null set, by Bolzano-Weierstrass theorem.

Examples

1. Let $f(x) = \operatorname{sgn} x,\ x \in [-2,2]$

Then
$$f(x) = -1,\quad -2 \le x < 0$$
$$= 0,\quad x = 0$$
$$= 1,\quad 0 < x \le 2$$

$f$ is bounded on $[-2,2]$ since $|f(x)| \le 1$ for all $x \in [-2,2]$. $f$ is continuous on $[-2,2]$ except at only one point $x=0$. Therefore $f$ is integrable on $[-2,2]$

2. Let $f(x) = [x],\ x \in [0,2]$

Then
$$f(x) = 0,\quad 0 \le x < 1$$
$$= 1,\quad 1 \le x < 2$$
$$= 2,\quad x = 2$$

$f$ is bounded on $[0,2]$, since $|f(x)| \le 2\ \forall x \in [0,2]$.
$f$ is continuous on $[0,2]$ except for the points $1, 2$.
So, $f$ is integrable on $[0,2]$

3. Let $f$ be defined on $[0,1]$ by $f(0) = 0$ and
$f(x) = \frac{1}{2^{n-1}},\ \frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$, for $n = 1, 2, 3, \ldots$

Here $f$ is monotone increasing and bounded in $[0,1]$. Therefore $f$ is integrable on $[0,1]$.

4. Let $f$ be defined on $[0,1]$ by $f(0) = 0$ and $f(x) = (-1)^{r-1}$, $\frac{1}{r+1} < x \le \frac{1}{r}$, for $r = 1,2,3,\cdots$

$f$ is continuous on $[0,1]$ except at the points $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$. The set of points of discontinuities of $f$ has only one limit point. Also $f$ is bounded on $[0,1]$. Therefore $f$ is integrable on $[0,1]$

5. A function $f$ is defined on $[0,1]$ by $f(0)=0$ and $f(x) = 0$, if $x$ be irrational

$= \frac{1}{q}$, if $x = \frac{p}{q}$ where $p, q$ are positive integers prime to each other

Show that $f$ is integrable on $[0,1]$ and show that $\int_0^1 f(x)\,dx = 0$

Proof: Here $f$ is bounded on $[0,1]$

Let us choose an $\varepsilon > 0$ such that $0 < \varepsilon < 2$

Then $\exists$ a natural number $k$ such that

$$k < \frac{2}{\varepsilon} \le k+1,$$ by Archimedean property of $\mathbb{R}$.

Let the rational numbers in $(0,1]$ be arranged as

$\frac{1}{1}; \frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{1}{4}, \frac{3}{4}; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \cdots; \frac{1}{k}, \cdots, \frac{k-1}{k}; \frac{1}{k+1}, \cdots, \frac{k}{k+1}; \cdots$

There are only a finite number of rationals of the form $p/q$ in $(0,1]$ with denominator $\le k$. At every such